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A COMPLEMENT TO THE PAPER "A POSITIVE ANSWER TO THE BASIS PROBLEM"

BY

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ABSTRACT

This complement includes stronger versions of the theorems of the paper "A positive answer to the basis problem", practically with the same proofs.

1. The existence of a uniformly minimal basis with fixed brackets and quasi-fixed permutations

This section is a continuation of the Introduction of [T]. We say that a sequence $\{x_n\}$, with $\{x_n, x_n^*\}$ biorthogonal, is a

(D_8) **Uniformly minimal basis with fixed brackets and quasi-fixed permutations** of a Banach space X, if $\{x_n\}$ and $\{x_n^*\}$ are both bounded, moreover there exist two fixed increasing sequences $\{p(m)\}$ and $\{q(m)\}$ of positive integers, with $q(m) < p(m) < q(m+1)$ for each m, such that, setting $p(0) = q(0) = 0$, every element x_0 of X has the following representation:

$$x_0 = \sum_{m=0}^{\infty} \sum_{n=q(m)+1}^{q(m+1)} x_{\pi(n)}^*(x_0) x_{\pi(n)},$$

where, for each m, $\{\pi(n)\}_{n=p(m)+1}^{p(m+1)}$ is a permutation of $\{n\}_{n=p(m)+1}^{p(m+1)}$ which depends on x_0.

The following theorem improves Theorem I of the Introduction.

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THEOREM 1: *Every separable Banach space has a uniformly minimal basis with fixed brackets and quasi-fixed permutations.*

An analogous improvement holds for Theorem II of the Introduction: this improvement follows from Theorem 1 by means of the same procedure as in §4 and §5.

Comment 6: It is impossible to improve Theorem 1 by omitting in (D_8) the finite block permutations, since the basis with fixed brackets does not exist in general. We also point out that the properties of (D_8) are hereditary for subsequences.

Proof of Theorem 1: The following minor modifications of the proof of [T] are needed: Follow the original proof in Section 3 of [T] up to (3.2). In the first, fifth and sixth lines of (3.2) replace q_{m+1} by $q(m)$ and, in the first, second and third lines of (3.2), replace t_{m+1} by $t'(m)$. On page 88, line 9, we replace $q_{m+1} = r_{t(m,i)}$ by $q(m) = r_{t(m,i)}$ for $i = r_{t(m)+1} - r_{t(m)}$. Note that the first part of (3.2) is satisfied. At this point we can choose by (2.17) a positive integer $t(m+1)$ such that, setting $q_{m+1} = r_{t(m+1)}$ (we use the subsequences of (2.17)),

$$\text{(3.10)}' \qquad \{\{u_{b(m),j,n}\}_{n=1}^{2^{b(m)}}\}_{j=1}^{R_{b(m)}} \subset \{x_n\}_{n=q(m)+1}^{q_{m+1}} \quad \text{with } b(m) > q(m).$$

Now we set $y_n = x_n$ and $y_n^* = x_n^*$ for $q(m)+1 \le n \le q_{m+1}$. Then we continue the proof of Theorem I, starting from line 11 on page 88.

We follow the proof of [T] up to the first line of page 93. Then we replace the paragraph starting with "Concluding case (A)" and ending with condition (3.18) by the following paragraph:

"By (2.17) and (3.10)$'$, there exists another index $j'(m')$ with $1 \le j'(m') \le R_{b(m')}$ such that

$$\text{(3.10)}'' \qquad \sum_{n=1}^{2^{b(m')}} |u^*_{b(m'),j'(m'),n}(x_0)| < \frac{1}{2^{b(m')}}.$$

Therefore, setting $0(m') = q(m') - \{r_{t(m')} + i(m') - 1 + r_{t(m'),i(m')-1} - r_{t(m')+1} + 2^{f(m',i(m'),s(m'),k(m'))}\}$, we can add a fourth term in (3.15), between the brackets, getting

$$\text{(3.15)}' \qquad \left\| x_0 - \left\{ \cdots + \sum_{n=1}^{0(m')} u^*_{b(m'),j'(m'),n}(x_0) u_{b(m'),j'(m'),n} \right\} \right\| < \frac{1}{2^m} + \frac{1}{2^{b(m')}}.$$

Concluding case (A), we call again $\{m(k)\}$ our subsequence. Then the preceding procedure gives the existence, for each k, of a permutation

$$\text{(3.18)} \qquad \{\pi(n)\}_{n=q_{m(k)}+1}^{q_{m(k)+1}} \text{ of } \{n\}_{n=q_{m(k)}+1}^{q_{m(k)+1}} \text{ with a positive number } \varepsilon'_{m(k)} \to 0$$

with k such that

$$\left\| x_0 - \left\{ \sum_{n=1}^{q_{m(k)}} y_n^*(x_0) y_n + \sum_{n=q_{m(k)}+1}^{q(m_k)} y_{\pi(n)}^*(x_0) y_{\pi(n)} \right\} \right\| < \varepsilon'_{m(k)}."$$

Then we continue the proof of Theorem I, starting with line 7 of page 93 up to and including (3.19). Following (3.19) we continue with the following part, which concludes the proof of Theorem 1:

"On the other hand, again by (2.17) and (3.10)$'$, there exists $j'(m'(k))$ with $1 \leq j'(m'(k)) \leq R_{b(m'(k))}$, such that we have (3.10)$''$ again, with m' replaced by $m'(k)$. Therefore, if $\{\pi(n)\}_{n=q_{m'(k)}+1}^{q_{m'(k)+1}}$ is any permutation of $\{n\}_{n=q_{m'(k)}+1}^{q_{m'(k)+1}}$ such that $\{y_{\pi(n)}\}_{n=q_{m'(k)}+1}^{q(m'(k))} = \{u_{b(m'(k)),j'(m'(k)),n}\}_{n=1}^{q(m'(k))-q_{m'(k)}}$, it again follows that

$$\left\| x_0 - \left\{ \sum_{n=1}^{q_{m'(k)}} y_n^*(x_0) y_n + \sum_{n=q_{m'(k)}+1}^{q(m'(k))} y_{\pi(n)}^*(x_0) y_{\pi(n)} \right\} \right\| < \varepsilon'_{m'(k)} + \frac{1}{2^{b(m'(k))}}."$$

Setting $p(m) = q_m$ for each m, this completes the proof of Theorem 1.

Reference

[T] P. Terenzi, *A positive answer to the basis problem*, Israel Journal of Mathematics **104** (1998), 51–124

CORRECTIONS TO THE PAPER "A POSITIVE ANSWER TO THE BASIS PROBLEM"

1. Page 56 bottom line: replace $s(m)$ by $r_{s(m)}$.
2. Page 62 line 8: replace "$\{g_n\}$ basic" by "$\{g_n\}$ of X^* such that $\{y_n, f_n\}$ and $\{y_n, g_n\}$ are biorthogonal, with $\{f_n\}$ bounded and $\{g_n\}$ basic".
3. Page 64 lines 20 and 22: replace "$(b) \Leftrightarrow (c)$" and "$(d) \Leftrightarrow (e)$" by "$(b) \Rightarrow (c)$" and "$(d) \Leftarrow (e)$" respectively.
4. Page 65 line 2: replace "$(b) \Rightarrow (d)$" by "$(b) \Leftrightarrow (d)$".
5. Page 66 line 16: replace "$e'-$" by "$\varepsilon'-$".
6. Page 67 line 19: replace "M" by "m".
7. Page 68 line 3: replace "T" by "$\perp$".
8. Page 72 line 8: replace "$q(1,q)$" by "$g(1,q)$".
9. Page 73 line 24: replace "$|a_{n'_k} l$" by "$|a_{n'_k}|$".
10. Page 76 line 16: replace "$2(1+\varepsilon') = 2\varepsilon'$" by "$2(1+\varepsilon') + 2\varepsilon'$".